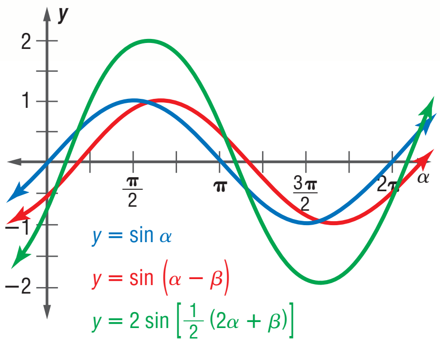
14-5 Sum and Differences of Angles Formulas

Main Ideas

- Find values of sine and cosine involving sum and difference formulas.
- Verify identities by using sum and difference formulas.

GET READY for the Lesson

Have you ever been talking on a cell phone and temporarily lost the signal? Radio waves that pass through the same place at the same time cause interference. *Constructive interference* occurs when two waves combine to have a greater amplitude than either of the component waves. *Destructive interference* occurs when the component waves combine to have a smaller amplitude.



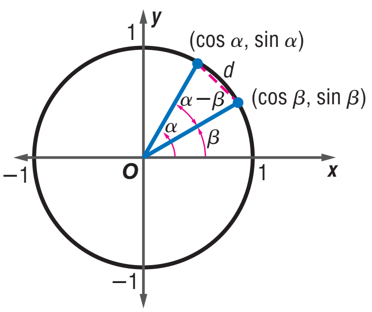
Sum and Difference Formulas Notice that the third equation shown above involves the sum of α and β. It is often helpful to use formulas for the trigonometric values of the difference or sum of two angles. For example, you could find sin 15° by evaluating sin (60° − 45°). Formulas can be developed that can be used to evaluate expressions like $\sin(\alpha - \beta)$ or $\cos(\alpha + \beta)$.

Reading Math

Greek Letters The Greek letter *beta*, β, can be used to denote the measure of an angle.

It is important to realize that $\sin(\alpha \pm \beta)$ is not the same as $\sin \alpha \pm \sin \beta$.

The figure at the right shows two angles α and β in standard position on the unit circle. Use the Distance Formula to find d, where $(x_1, y_1) = (\cos \beta, \sin \beta)$ and $(x_2, y_2) = (\cos \alpha, \sin \alpha)$.



$$d = \sqrt{(\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2}$$

$$d^2 = (\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2$$

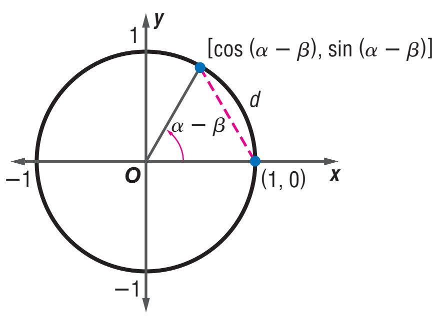
$$d^2 = (\cos^2 \alpha - 2\cos \alpha \cos \beta + \cos^2 \beta) + (\sin^2 \alpha - 2\sin \alpha \sin \beta + \sin^2 \beta)$$

$$d^2 = \cos^2 \alpha + \sin^2 \alpha + \cos^2 \beta + \sin^2 \beta - 2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta$$

$$d^2 = 1 + 1 - 2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta \quad \sin^2 \alpha + \cos^2 \alpha = 1 \text{ and}$$

$$d^2 = 2 - 2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta \quad \sin^2 \beta + \cos^2 \beta = 1$$

Now find the value of d^2 when the angle having measure $\alpha - \beta$ is in standard position on the unit circle, as shown in the figure at the left.



$$d = \sqrt{[\cos(\alpha - \beta) - 1]^2 + [\sin(\alpha - \beta) - 0]^2}$$

$$\begin{aligned} d^2 &= [\cos(\alpha - \beta) - 1]^2 + [\sin(\alpha - \beta) - 0]^2 \\ &= [\cos^2(\alpha - \beta) - 2\cos(\alpha - \beta) + 1] + \sin^2(\alpha - \beta) \\ &= \cos^2(\alpha - \beta) + \sin^2(\alpha - \beta) - 2\cos(\alpha - \beta) + 1 \\ &= 1 - 2\cos(\alpha - \beta) + 1 \\ &= 2 - 2\cos(\alpha - \beta) \end{aligned}$$

By equating the two expressions for d^2, you can find a formula for $\cos(\alpha - \beta)$.

$$d^2 = d^2$$

$$2 - 2\cos(\alpha - \beta) = 2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta$$

$$-1 + \cos(\alpha - \beta) = -1 + \cos\alpha\cos\beta + \sin\alpha\sin\beta \quad \text{Divide each side by } -2.$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \quad \text{Add 1 to each side.}$$

Use the formula for $\cos(\alpha - \beta)$ to find a formula for $\cos(\alpha + \beta)$.

$$\cos(\alpha - \beta) = \cos[\alpha - (-\beta)]$$

$$= \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta)$$

$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta \quad \cos(-\beta) = \cos\beta;\ \sin(-\beta) = -\sin\beta$$

You can use a similar method to find formulas for $\sin(\alpha + \beta)$ and $\sin(\alpha - \beta)$.

KEY CONCEPT *Sum and Difference of Angles Formulas*

The following identities hold true for all values of α and β.

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

Notice the symbol $\mp$ in the formula for $\cos(\alpha \pm \beta)$. It means "minus or plus." In the cosine formula, when the sign on the left side of the equation is plus, the sign on the right side is minus; when the sign on the left side is minus, the sign on the right side is plus. The signs match each other in the sine formula.

EXAMPLE 1 Use Sum and Difference of Angles Formulas

Find the exact value of each expression.

a. cos 75°

Use the formula $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

$$\cos 75° = \cos(30° + 45°) \quad \alpha = 30°, \beta = 45°$$

$$= \cos 30°\cos 45° - \sin 30°\sin 45°$$

$$= \left(\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\cdot\frac{\sqrt{2}}{2}\right) \quad \text{Evaluate each expression.}$$

$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \quad \text{Multiply.}$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4} \quad \text{Simplify.}$$

b. sin (−210°)

Use the formula $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

$$\sin(-210°) = \sin(60° - 270°) \quad \alpha = 60°, \beta = 270°$$

$$= \sin 60°\cos 270° - \cos 60°\sin 270°$$

$$= \left(\frac{\sqrt{3}}{2}\right)(0) - \left(\frac{1}{2}\right)(-1) \quad \text{Evaluate each expression.}$$

$$= 0 - \left(-\frac{1}{2}\right) \text{ or } \frac{1}{2} \quad \text{Simplify.}$$

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CHECK Your Progress

1A. sin 15° **1B.** cos (−15°)

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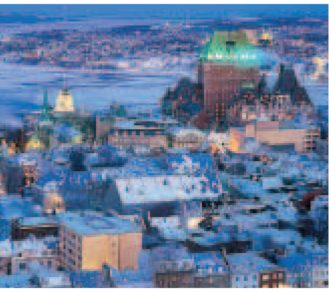
Reading Math

Greek Letters The symbol ϕ is the lowercase Greek letter *phi*.



Real-World Link In the northern hemisphere, the day with the least number of hours of daylight is December 21 or 22, the first day of winter.

Source: www.infoplease.com

Real-World EXAMPLE

2 **PHYSICS** **On June 22, the maximum amount of light energy falling on a square foot of ground at a location in the northern hemisphere is given by E sin (113.5° − ϕ), where ϕ is the latitude of the location and E is the amount of light energy when the Sun is directly overhead. Use the difference of angles formula to determine the amount of light energy in Rochester, New York, located at a latitude of 43.1° N.**

Use the difference formula for sine.

$$\begin{aligned} \sin(113.5° - \phi) &= \sin 113.5° \cos \phi - \cos 113.5° \sin \phi \\ &= \sin 113.5° \cos 43.1° - \cos 113.5° \sin 43.1° \\ &= 0.9171 \cdot 0.7302 - (-0.3987) \cdot 0.6833 \\ &= 0.9420 \end{aligned}$$

In Rochester, New York, the maximum light energy per square foot is 0.9420E.

CHECK Your Progress

2. Determine the amount of light energy in West Hollywood, California, which is located at a latitude of 34.1° N.

Verify Identities You can also use the sum and difference formulas to verify identities.

EXAMPLE Verify Identities

3 **Verify that each of the following is an identity.**

a. **sin (180° + θ) = −sin θ**

$$\begin{aligned} \sin(180° + \theta) &\stackrel{?}{=} -\sin\theta && \text{Original equation} \\ \sin 180° \cos\theta + \cos 180° \sin\theta &\stackrel{?}{=} -\sin\theta && \text{Sum of angles formula} \\ 0 \cos\theta + (-1)\sin\theta &\stackrel{?}{=} -\sin\theta && \text{Evaluate each expression.} \\ -\sin\theta &= -\sin\theta && \text{Simplify.} \end{aligned}$$

b. **cos (180° + θ) = −cos θ**

$$\begin{aligned} \cos(180° + \theta) &\stackrel{?}{=} -\cos\theta && \text{Original equation} \\ \cos 180° \cos\theta - \sin 180° \sin\theta &\stackrel{?}{=} -\cos\theta && \text{Sum of angles formula} \\ (-1)\cos\theta - 0 \sin\theta &\stackrel{?}{=} -\cos\theta && \text{Evaluate each expression.} \\ -\cos\theta &= -\cos\theta && \text{Simplify.} \end{aligned}$$

CHECK Your Progress

3A. sin (90° − θ) = cos θ **3B.** cos (90° + θ) = −sin θ

CHECK Your Understanding

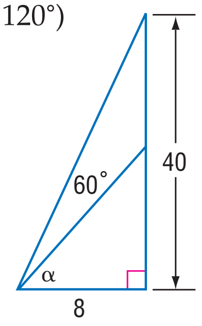
Example 1 (pp. 849–850)

Find the exact value of each expression.

1. $\sin 75°$
2. $\sin 165°$
3. $\cos 255°$
4. $\cos(-30°)$
5. $\sin(-240°)$
6. $\cos(-120°)$

Example 2 (p. 850)

7. **GEOMETRY** Determine the exact value of $\tan \alpha$ in the figure.



Example 3 (p. 850)

Verify that each of the following is an identity.

8. $\cos(270° - \theta) = -\sin \theta$
9. $\sin\left(\theta + \frac{\pi}{2}\right) = \cos \theta$
10. $\sin(\theta + 30°) + \cos(\theta + 60°) = \cos \theta$

Exercises

HOMEWORK HELP

For Exercises	See Examples
11–24	1
25–28	2
29–36	3

Find the exact value of each expression.

11. $\sin 135°$
12. $\cos 105°$
13. $\sin 285°$
14. $\cos 165°$
15. $\cos 195°$
16. $\sin 255°$
17. $\cos 225°$
18. $\sin 315°$
19. $\sin(-15°)$
20. $\cos(-45°)$
21. $\cos(-150°)$
22. $\sin(-165°)$

PHYSICS **For Exercises 23–26, use the following information.**

On December 22, the maximum amount of light energy that falls on a square foot of ground at a certain location is given by $E \sin(113.5° + \phi)$, where ϕ is the latitude of the location. Find the amount of light energy, in terms of E, for each location.

23. Salem, OR (Latitude: 44.9° N)
24. Chicago, IL (Latitude: 41.8° N)
25. Charleston, SC (Latitude: 28.5° N)
26. San Diego, CA (Latitude 32.7° N)

Verify that each of the following is an identity.

27. $\sin(270° - \theta) = -\cos \theta$
28. $\cos(90° + \theta) = -\sin \theta$
29. $\cos(90° - \theta) = \sin \theta$
30. $\sin(90° - \theta) = \cos \theta$
31. $\sin\left(\theta + \frac{3\pi}{2}\right) = -\cos \theta$
32. $\cos(\pi - \theta) = -\cos \theta$
33. $\cos(2\pi + \theta) = \cos \theta$
34. $\sin(\pi - \theta) = \sin \theta$

COMMUNICATION **For Exercises 35 and 36, use the following information.**

A radio transmitter sends out two signals, one for voice communication and another for data. Suppose the equation of the voice wave is $v = 10 \sin(2t - 30°)$ and the equation of the data wave is $d = 10 \cos(2t + 60°)$.

35. Draw a graph of the waves when they are combined.
36. Refer to the application at the beginning of the lesson. What type of interference results? Explain.

EXTRA PRACTICE
See pages 923, 939.

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Verify that each of the following is an identity.

37. $\sin(60° + \theta) + \sin(60° - \theta) = \sqrt{3} \cos \theta$
38. $\sin\left(\theta + \frac{\pi}{3}\right) - \cos\left(\theta + \frac{\pi}{6}\right) = \sin \theta$
39. $\sin(\alpha + \beta) \sin(\alpha - \beta) = \sin^2 \alpha - \sin^2 \beta$
40. $\cos(\alpha + \beta) = \dfrac{1 - \tan \alpha \tan \beta}{\sec \alpha \sec \beta}$

H.O.T. Problems

41. OPEN ENDED Give a counterexample to the statement that $\sin(\alpha + \beta) = \sin \alpha + \sin \beta$ is an identity.

42. REASONING Determine whether $\cos(\alpha - \beta) < 1$ is *sometimes*, *always*, or *never* true. Explain your reasoning.

43. CHALLENGE Use the sum and difference formulas for sine and cosine to derive formulas for $\tan(\alpha + \beta)$ and $\tan(\alpha - \beta)$.

44. *Writing in Math* Use the information on page 848 to explain how the sum and difference formulas are used to describe communication interference. Include an explanation of the difference between constructive and destructive interference.

STANDARDIZED TEST PRACTICE

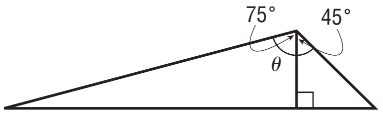
45. ACT/SAT Find the exact value of $\sin \theta$.

A $\frac{\sqrt{3}}{2}$

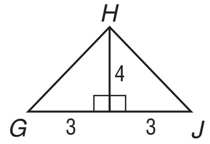
B $\frac{\sqrt{2}}{2}$

C $\frac{1}{2}$

D $\frac{\sqrt{3}}{3}$



46. REVIEW Refer to the figure below. Which equation could be used to find $m\angle G$?



F $\sin G = \frac{3}{4}$

G $\cos G = \frac{3}{4}$

H $\cot G = \frac{3}{4}$

J $\tan G = \frac{3}{4}$

Spiral Review

Verify that each of the following is an identity. (Lesson 14-4)

47. $\cot \theta + \sec \theta = \frac{\cos^2 \theta + \sin \theta}{\sin \theta \cos \theta}$

48. $\sin^2 \theta + \tan^2 \theta = (1 - \cos^2 \theta) + \frac{\sec^2 \theta}{\csc^2 \theta}$

49. $\sin \theta (\sin \theta + \csc \theta) = 2 - \cos^2 \theta$

50. $\frac{\sec \theta}{\tan \theta} = \csc \theta$

Simplify each expression. (Lesson 14-3)

51. $\frac{\tan \theta \csc \theta}{\sec \theta}$

52. $4\left(\sec^2 \theta - \frac{\sin^2 \theta}{\cos^2 \theta}\right)$

53. $(\cot \theta + \tan \theta)\sin \theta$

54. $\csc \theta \tan \theta + \sec \theta$

55. AVIATION A pilot is flying from Chicago to Columbus, a distance of 300 miles. In order to avoid an area of thunderstorms, she alters her initial course by 15° and flies on this course for 75 miles. How far is she from Columbus? (Lesson 13-5)

56. Write $6y^2 - 34x^2 = 204$ in standard form. (Lesson 10-6)

GET READY for the Next Lesson

PREREQUISITE SKILL **Solve each equation.** (Lesson 5-5)

57. $x^2 = \frac{20}{16}$

58. $x^2 = \frac{9}{25}$

59. $x^2 = \frac{5}{25}$

60. $x^2 = \frac{18}{32}$